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Modules without Self-Extensions over Radical Cube Zero Rings

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A conjecture of Tachikawa states that every finitely generated non-projective module M over a self-injective artinian ring R has a self-extension, i.e., $\mathrm{Ext}^i_R(M, M) \neq 0$ for some $i \geqslant 1$. We show that Tachikawa's conjecture holds for a class of radical cube zero rings.

1. INTRODUCTION AND MAIN RESULT

Let R be a QF ring (=self-injective artinian ring), and let all R-modules under consideration be finitely generated right R-modules. Let J denote the Jacobson radical of R.

A long-standing conjecture of Tachikawa [3, p. 116] states that every non-projective module M has $\mathrm{Ext}^i_R(M, M) \neq 0$ for some $i \geqslant 1$. By use of completely different methods, this conjecture has been answered in the affirmative for several classes of rings, including group algebras of finite p-groups over a field [3, Theorem 8.6] (this result has later been generalized to arbitrary finite groups [2, Proposition 3.2]) and symmetric self-injective algebras with $J^3 = 0$. For local QF Artin algebras, a stronger results holds:

THEOREM [1, Theorem 3.4]. *Let R be a local QF Artin algebra with $J^3 = 0$. Then, for every non-projective R-module M, one has* $\mathrm{Ext}^1_R(M, M) \neq 0$.

The proof of this theorem relies on the assumption that R is an Artin algebra and makes use of the stable Auslander–Reiten quiver associated with R.

Let us introduce the following notation. For a two-sided vector space V over a division ring D, and for a vector $x \in V$, let $C_D(x)$ be the centralizer of x in D. We can now state our main result.

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THEOREM. *Let R be a local QF ring with $J^3=0$. Let $D=R/J$. Assume that for every $x \in J/J^2$ the right index of $C_D(x)$ in D is finite. Then, for every non-projective R-module M, one has* $\operatorname{Ext}^1_R(M, M) \neq 0$.

Note that the finiteness condition of the theorem is satisfied if R is an Artin algebra. More rings sharing this condition are constructed in Section 3. In the proof of the theorem, we will make use of the fact that Tachikawa's conjecture can be rewritten as a matrix problem. The proof will then follow from elementary matrix arguments. Some other consequences of the matrix approach to Tachikawa's conjecture will be given in Section 3.

2. PROOF OF THE THEOREM

Let a complete minimal injective-projective resolution

$$\cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \longrightarrow \cdots$$

of M be given, with $M=\operatorname{Im} d_0$ and $K_i=\operatorname{Ker} d_i$ for $i \in \mathbb{Z}$. By a well-known dimension shift argument valid for QF rings R, the group $\operatorname{Ext}^1_R(M, M)$ is isomorphic to $\operatorname{Ext}^1_R(K_i, K_i)$ for every $i \in \mathbb{Z}$. Hence, by shifting from $M=K_{-1}$ to an index i with P_i of minimal rank (note that all P_i's are free) and by renaming the indices, we may assume that the rank of P_0 is less than or equal to the ranks of P_1 and P_{-1}. The group $\operatorname{Ext}^1_R(M, M)$ equals $\operatorname{Hom}_R(K_0, M)$ modulo the subgroup U of maps which factorize over P_0.

Let any $f \in \operatorname{Hom}_R(K_0, M)$ be given and consider the following diagram.

$$\begin{tikzcd}
R^m \arrow[rr, "d_1"] \arrow[dr] \arrow[dd, "h"'] & & R^n \arrow[dd, "g"] \\
& K_0 \arrow[ur] \arrow[dd, "f"] & \\
R^n \arrow[rr, "d_0"] \arrow[dr] & & R^k \\
& M \arrow[ur] &
\end{tikzcd}$$

Here, we are setting $P_1=R^m$, $P_0=R^n$, and $P_{-1}=R^k$. Note that $n \leqslant m$ and $n \leqslant k$. The map f can be lifted to h and, by injectivity of R, to g. Thus f determines a $k \times m$ matrix $d_0 h = g d_1 \in d_0 R^{n \times m} \cap R^{k \times n} d_1$. (Here, $R^{i \times j}$ denotes the set of all $i \times j$ matrices over R.) Conversely, every such matrix determines a map from K_0 to M, by multiplication from the left. This establishes a 1–1 correspondence

$$\alpha: \operatorname{Hom}_R(K_0, M) \to d_0 R^{n\times m} \cap R^{k\times n} d_1.$$

Under this correspondence, the subgroup V of $\operatorname{Hom}_R(K_0, M)$ given by $V = \operatorname{Hom}_R(K_0, \operatorname{Soc} M) \cong \operatorname{Hom}_R(K_0/K_0 J, \operatorname{Soc} M)$ goes to $\alpha(V) = (sD)^{m\times k}$, where s is some element in R generating $\operatorname{Soc} R$. Under the same correspondence, the subgroup U of $\operatorname{Hom}_R(K_0, M)$ goes to $d_0 R^{n\times n} d_1$. This follows from the fact that a factorization of f over the inclusion $K_0 \subset R^n$ can be factorized to a map from the right upper corner to the left lower corner of the diagram and then has the form $d_0 a d_1$, for some matrix $a \in R^{n\times n}$. The claim $\operatorname{Ext}^1_R(M, M) \neq 0$ is hence equivalent to $d_0 R^{n\times n} d_1 \subsetneq d_0 R^{n\times m} \cap R^{k\times n} d_1$. It is sufficient to show that $\alpha(U) \subsetneq \alpha(V)$.

Consider now the map $\beta: D^{n\times n} \to sD^{k\times m}$, defined as follows. Given $\bar{a} \in D^{n\times n}$, lift $\bar{a}$ to a matrix $a \in R^{n\times n}$ and form $d_0 a d_1$. Since, by minimality of the projective resolution of M, d_0 and d_1 both have all their entries in J, the matrix $d_0 a d_1$ is in $(\operatorname{Soc} R)^{m\times k}$, and $d_0 a d_1$ does not depend on the choice of the lift from $\bar{a}$ to a. Obviously, one has $\alpha(U) = d_0 R^{n\times n} d_1 = \operatorname{Im} \beta \subset \alpha(V)$. We will show that the inclusion is proper.

For every entry x of the matrix d_1, let $x = x + J^2 \in J/J^2$, and let $L = \bigcap_x C_D(x)$, x ranging over all entries of d_1. By the finiteness condition of the theorem, the right index of L in D is finite. Note that β is then right L-linear, and that $\operatorname{Ker} \beta$ is non-trivial since it contains the identity matrix. Looking at right L-dimensions, we then have $\dim(\operatorname{Im} \beta) = [D:L]\, n^2 - \dim(\operatorname{Ker} \beta) \leqslant [D:L]\, km - 1 < [D:L]\, km = \dim(\alpha(V))$. This completes the proof.

3. Remarks

(1) We will construct a division ring D with the following property: For every $x \in D$, the centralizer $C_D(x)$ of x in D has finite index in D, but D has infinite dimension over its center.

Let K be a field with $\operatorname{char}(K) \neq 2$ and let $A = K[X_1, X_2, \ldots Y_1, Y_2, \ldots]$ be a polynomial ring over K with the following multiplication: All X's commute with each other and with K, all Y's commute with each other and with K, and

$$Y_i X_j = \begin{cases} X_j Y_i & \text{if } i \neq j \\ -X_i Y_i & \text{if } i = j. \end{cases}$$

Then A is a domain. For every n, the subring A_n in the variables up to index n only, is noetherian and hence an Ore domain. Since every element in A has finite support, A is an Ore domain, as well. Denote the field of fractions of A by D. We will show that D has the desired property. Let $x \in D$. Again, x has finite support and commutes with almost all of the variables, and x commutes with the squares of the finitely many remaining variables. Then D, as a right vector space over $C_D(x)$, has finite dimension. (Let $f+gX$ be a denominator. Then $(f+gX)(h+kX)=fh+fkX+gXh+gXkX=fh+gk'X^2+(fk+gh')X$, for some k' and h'. The Ore condition guarantees that $fk+gh'=0$ for suitable k, g', this means that odd powers of X can be removed from the denominator.) D has infinite dimension over its center which is the polynomial ring over the squares of all the variables.

(2) With D from (1), let $R=D[T,S]/(T^2,S^2)$ in variables T, S which commute with each other and with D. Then R is a local QF ring with $J^3=0$, and R is not an Artin algebra.

(3) Let R be a (local) commutative QF ring with J of arbitrary nilpotency index. Given M, we may define the complete homology $\mathrm{Ext}^i_R(M,M)$ for all $i \in \mathbf{Z}$, this group to be equal to $\mathrm{Hom}_R(K_{i-1},M)$ modulo the subgroup of factorizable maps.

CLAIM. *If M is non-projective, then* $\mathrm{Ext}_R^{-1}(M,M) \neq 0$. (*This does not answer Tachikawa's conjecture for commutative rings, but it shows at least that M does not have trivial positive and negative homology.*)

Proof. Using the notation of Section 2 and looking at the index -1 in the complete resolution of M, we see that every map from K_{-2} to M which factorizes over P_{-2} induces a matrix d_0ad_{-1}. For the trace, we obtain $\mathrm{Tr}(d_0ad_{-1})=\mathrm{Tr}(d_{-1}d_0a)=\mathrm{Tr}(0)=0$. But there are maps from K_{-2} to M which have non-zero trace, such as the one which maps the first summand of $K_{-2}/K_{-2}J$ onto the first summand of Soc M and which kills the rest of K_{-2}. The matrix for this map has entry s in position $(1,1)$ and 0 elsewhere.

(4) Let R be as in Remark (3) and let M be a cyclic non-projective module. Then, with the notation from Section 2, $n=1$, hence $a \in R$ and $d_0ad_1=ad_0d_1=0$ for every factorizable map from K_0 to M. Since $\mathrm{Hom}_R(K_0,M)$ is obviously non-zero, one has $\mathrm{Ext}^1_R(M,M) \neq 0$.

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